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LETTERS TO THE EDITOR

FUNDAMENTAL MODES OF A CIRCULAR MEMBRANE WITH RADIAL CONSTRAINTS ON THE BOUNDARY

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1. INTRODUCTION

The vibration of circular membranes has been expounded by Rayleigh [1]. He also considered the almost circular membrane, where the boundary has been perturbed slightly. The present note studies the circular membrane with additional radial constraints on the boundary (Figure 1(a)). Since the perturbations are not small, Rayleigh's technique cannot be used. Instead we shall use a two-region eigenfunction expansion and matching method. This method is suitable for some complex geometries and has been applied previously to coaxial waveguides [2,3].

The governing equation is

$$w_{rr} + \frac{1}{r}\,w_r + \frac{1}{r^2}\,w_{\theta\theta} + k^2 w = 0, \tag{1}$$

where all lengths have been normalized by the radius R of the membrane and $k =$ (frequency) R (tension per length/mass per area)$^{1/2}$. The boundary conditions are that w is zero on the circle $r = 1$ and also on the M equally-spaced constraints of length b.

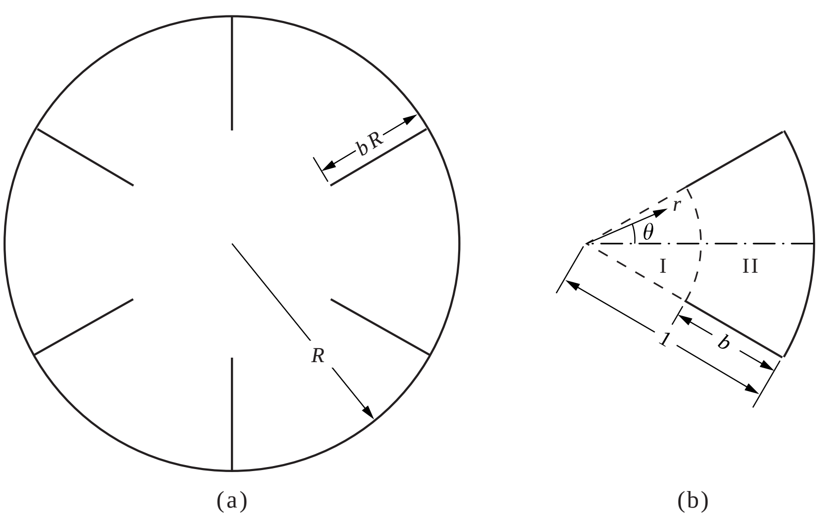

Figure 1. (a) The circular membrane with M equally-spaced radial constraints. (b) A repeating sector separated into two regions.

2. SOLUTION

For the fundamental mode, w has M-fold symmetry and we need to consider only the sector $|\theta| \leqslant \beta = \pi/M$. The sector is further partitioned into regions I and II (Figure 1(b)). For region I the general solution to equation (1) which is even in $\theta = 0$ and $\theta = \beta$ and satisfies boundedness at $r = 0$ is

$$w_I(r, \theta) = A_0 J_0(kr) + \sum_1^{\infty} A_n (nM)! \cos(nM\theta) J_{nM}(kr). \tag{2}$$

Here J, Y are Bessel functions and the factor $(nM)!$ is included to ensure the coefficients A_n to be of reasonable magnitudes. The general solution for region II which is even in θ and zero on $r = 1$ and $\theta = \beta$ is

$$w_{II}(r, \theta) = \sum_1^{\infty} B_n \cos(v_n \theta) H_n(r), \tag{3}$$

where $v_n = (n - \frac{1}{2})\pi/\beta$ and

$$H_n(r) \equiv Y_{v_n}(k) J_{v_n}(kr) - J_{v_n}(k) Y_{v_n}(kr). \tag{4}$$

Now w_I and w_{II} are to be continuous along their common boundary:

$$w_I(1 - b, \theta) = w_{II}(1 - b, \theta), \qquad \frac{\partial w_I}{\partial r}(1 - b, \theta) = \frac{\partial w_{II}}{\partial r}(1 - b, \theta). \tag{5, 6}$$

We truncate the series A_n to N terms and B_n to $N + 1$ terms. Integrating equation (5) from $\theta = 0$ to $\theta = \beta$ yields

$$\beta J_0(kc) A_0 + \sum_1^{N+1} \frac{(-1)^n}{v_n} H_n(c) B_n = 0, \tag{7}$$

where $c = 1 - b$. Multiplying by $\cos(mM\theta)$ and integrating yields

$$\frac{\beta}{2}(mN)! J_{mN}(kc) A_m - \sum_1^{N+1} G_{mn} H_n(c) B_n = 0, \qquad m = 1 \text{ to } N, \tag{8}$$

where

$$G_{mn} = \int_0^{\beta} \cos(v_n \theta) \cos(mM\theta)\, \mathrm{d}\theta$$

$$= \frac{1}{2}\left[\frac{\sin(v_n + mM)\beta}{v_n + mM} + \frac{\sin(v_n - mM)\beta}{v_n - mM}\right]. \tag{9}$$

TABLE 1

Fundamental frequency of sector with angle $2\pi/M$

M	1	2	3	4	5	6	8	10	12
k	3·142	3·832	4·493	5·136	5·763	6·380	7·588	8·771	9·936

Similarly, equation (6) gives

$$-\beta k J_1(kc)A_0 + \sum_1^{N+1} \frac{(-1)^n}{v_n} H_n'(c)B_n = 0, \tag{10}$$

$$\frac{\beta k}{4}[J_{mM-1}(kc) - J_{mM+1}(kc)](mN)!A_m - \sum_1^{N+1} G_{mn}H_n'(c)B_n = 0, \qquad m = 1 \text{ to } N. \tag{11}$$

Equations (7), (8), (10) and (11) represent $2N+2$ homogeneous equations and unknowns. For the non-trivial solution, the determinant of the coefficients is set to zero. This gives the characteristic equation which is solved for the minimum value of k. Accuracy is improved by increasing N. It was found that $N = 10$ is adequate for three significant digits in k. After k is determined, set $A_0 = 1$ and solve for the rest of the coefficients. The vibration mode w is then obtained.

3. RESULTS AND DISCUSSION

The characteristic equation is in closed form when $b = 0$ or $b = 1$. When $b = 0$ the geometry is the circular cylinder. The fundamental frequency is the first root of $J_0(k) = 0$, or $k = 2{\cdot}4048$. For $b = 1$ the constraints separate the circle into circular sectors. The fundamental frequency is the first root of $J_{M/2}(k) = 0$. See Table 1.

For $0 < b < 1$, the method in the previous section is used. Figure 2 shows the result. For given M, the fundamental frequency increases with b, starting from the circular value of 2·4048 and ending with the sector value given in Table 1. The higher the number of constraints M, the higher the frequency. For the $M \to \infty$ case the constraints effectively shrink the circular membrane to a smaller one with radius $1 - b$. The fundamental frequency is then

$$k = \frac{2{\cdot}4048}{1-b}. \tag{12}$$

One can also see from Figure 2 that for larger M, the frequency seems to have two distinct phases: a rising phase for small b, and a plateau phase for large b. Figure 3 shows typical amplitude curves for $M = 6$. Only the sector $0 \leqslant \theta \leqslant \beta$ is shown. The amplitude, of course, is unique up to a multiplicative constant. Figure 3(a) shows the mode for $b = 0{\cdot}5$ on the rising phase. The maximum amplitude is at $r = 0$, the center. The vibration mode is similar to that of a circular membrane, with peripheral regions periodically suppressed by the protruding constraints.

Figure 3(b) shows the mode for $b = 0{\cdot}75$, which is at the transition between the rising phase and the plateau phase. We see that the maximum amplitude shifted from the center to M points between the constraints (near $r = 0{\cdot}65$). The latter characteristic is a property of the sector membrane shown in Figure 3(c). (The exact solution is $w \sim \cos(3\theta)J_3(6{\cdot}380r)$.) In the plateau phase both the mode shape and the frequency are almost constant for the range $0{\cdot}8 < b < 1$.

The frequency of a circular membrane is increased by constraints on the boundary. The constraints may be unavoidable, or may also be used to actively alter the frequency. It was found that the frequency is most sensitive if the lengths of the constraints are in the range $0{\cdot}5 < b < 0{\cdot}7$.

Recently [4] a membrane strip with periodic boundary constraints was studied. In that problem the matching of two regions was unnecessary and eigenfunction

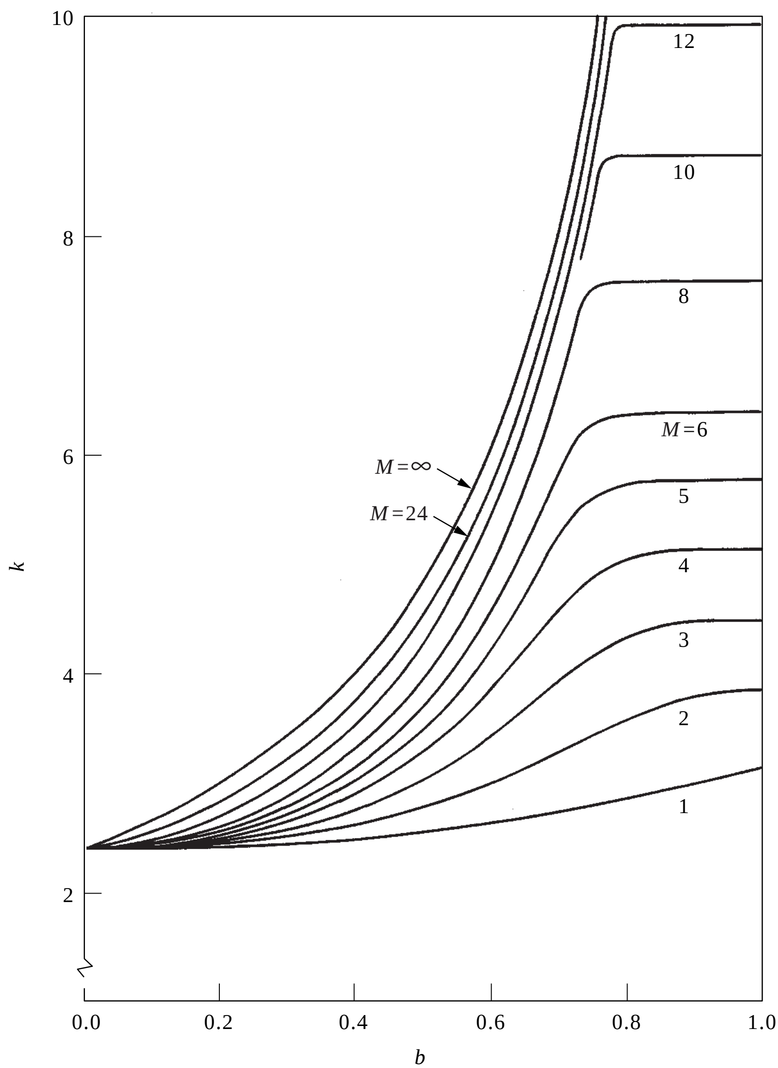

Figure 2. Fundamental frequency as a function of b for various M.

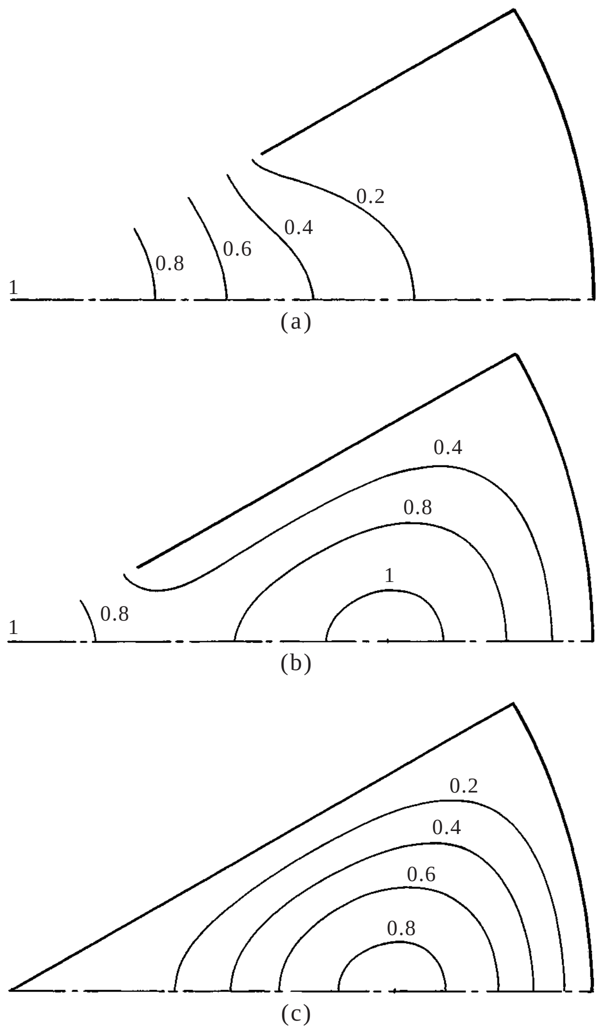

Figure 3. Vibrational mode for $M = 6$; (a) $b = 0{\cdot}5$, (b) $b = 0{\cdot}75$, (c) $b = 1$.

expansions in a single region sufficed. Also the distinct change in vibration character as in the present circular membrane was not found.

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